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Hydrodynamic limit for interacting Ornstein–Uhlenbeck particles

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Abstract

We consider a system of interacting Ornstein–Uhlenbeck particles moving in a d-dimensional torus. The interaction between particles is given by a short-range superstable pair potential V. We prove that, in a diffusive scaling limit, the density of particles satisfies a non-linear partial differential equation. This generalizes to higher dimensions a result of Olla and Varadhan (cf. (Comm. Math. Phys. 125 (1993) 523)). © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we study the hydrodynamic limit for interacting Ornstein–Uhlenbeck particles evolving on the d-dimensional torus. Our model is the following: we consider a system of N particles which evolve in time according to the following system of stochastic differential equations:

$$\mathrm{d}q_i(\tau) = v_i(\tau)\,\mathrm{d}\tau,$$

$$\mathrm{d}v_i(\tau) = -\sum_{j\neq i} 2\nabla V(q_i(\tau) - q_j(\tau))\,\mathrm{d}\tau - \frac{\gamma}{2}\,v_i(\tau)\,\mathrm{d}\tau + \sqrt{\frac{\gamma}{\beta}}\,\mathrm{d}W_i(\tau). \tag{1.1}$$

Here, $q_i(\tau)$ is the position of the ith particle at time τ. All the $\{q_i, i = 1,\dots,N\}$, are confined in a d-dimensional torus $\mathbb{T}_L^d$ of length L. $v_i(\tau)$ is the velocity of the ith particle at time τ. $\{W_i(\tau), i = 1,\dots,N\}$ are N independent standard Wiener processes, γ is the

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friction coefficient and β is the inverse of temperature. Moreover, by $\sum_{(j\neq i)}$ we will denote the sum over all unordered couples $(j\neq i)$, i.e. $\frac{1}{2}\sum_j\sum_{i\neq j}$.

The interaction between particles is given by the two body potential V. We assume that V is smooth, has a finite range and V is superstable (see Ruelle, 1969). The scaling is such that the range of the interaction is always of the same order as the inter-particles distance. Therefore, in a typical configuration, each particle interacts with only a finite number of particles at each time. Because we want the range of the interaction of the same order as the inter-particles distance, we take it of order L with $L\sim N^{1/d}\sim\varepsilon^{-1}$. Then, the hydrodynamic scaling limit will correspond to make ε going to 0.

Consider the diffusive rescaling of space and time: call $x_i^\varepsilon(\tau)=\varepsilon q_i(\varepsilon^{-2}\tau)$. Let ρ_0 : $\mathbb{T}^d\to[0,1]$ be an initial density profile and let

$$\alpha_\varepsilon(t,\mathrm{d}x)=\varepsilon^d\sum_i\delta_{x_i^\varepsilon(t)}(\mathrm{d}x)$$

be the empirical distribution of the process at time t defined in $\mathbb{T}^d$. We can see α_ε as an element of $\mathscr{C}\doteq\mathscr{C}(\mathbb{R}^+,\mathscr{M}(\mathbb{T}^d))$ the space of continuous functions on $\mathbb{R}^+$ with values in the space $\mathscr{M}(\mathbb{T}^d)$ of probability measures on $\mathbb{T}^d$ (endowed by the weak topology). If Q_ε is the distribution of α_ε on $\mathscr{C}$, we will prove that, as ε goes to zero, Q_ε will concentrate around a single trajectory of the form $\alpha(t,\mathrm{d}x)=\rho(t,x)\,\mathrm{d}x$ where $\rho(t,x)$ is the solution of the following non-linear bulk diffusion equation:

$$\partial_t\rho(t,x)=\frac{2}{\gamma}\,\Delta_x P(\rho(t,x)),$$

$$\rho(0,\cdot)=\rho_0(\cdot), \tag{1.2}$$

where $P(\rho)$ is the thermodynamic pressure expressed as a function of the density ρ.

In dimension $d=1$, this result was proven by Olla and Varadhan (1991). Nevertheless, methods used in Olla and Varadhan (1991) are only valid in one dimension. In this paper, we use the relative entropy method introduced by Yau (1991) which can be applied to continuous systems in dimension $d\geqslant 1$.

This method consists on considering the relative entropy and its rate of change with respect to local Gibbs states. Olla et al. (1993) adapted this method to an Hamiltonian system with weak noise and they proved that the conserved quantities of the system evolve according to the Euler equations in the time interval where the solutions of these equations are smooth.

As in Olla et al. (1993), the relative entropy method applied to our model (1.1) works only for smooth solutions of the macroscopic equation. For this reason, we have to take the initial profile of density ρ_0 inside the one-phase region, where the thermodynamic pressure P is a smooth function of the density. Then the maximum principle will keep the solution of (1.2) inside this region.

Butta and Lebowitz (1999), by using the relative entropy method, proved the hydrodynamic limit of Brownian particles in interaction with short- and long-range forces. This work generalizes in all dimensions the results of Varadhan (1991). The processes considered in Butta and Lebowitz (1999) and Varadhan (1991) are reversible whereas the Ornstein–Uhlenbeck process (1.1) is not. Moreover, the generator of the process

(1.1) is degenerate on its action on the positions of the particles whereas the generator in Butta and Lebowitz (1999) is an elliptic one.

Moreover, in order to make Yau's method work, we need to add a correction term of order one to the local Gibbs state associated to the macroscopic evolution $\rho(t,x)$ given by (1.2). This correction centers the "local Maxwellians" distribution of the velocities around the drift imposed by the gradient of the density.

The paper is organized as follows: in the next section, we present our macroscopic system of interacting particles and we introduce the thermodynamic functions useful to get the macroscopic equation (1.2). Then we state our results. In Section 3, we introduce the relative entropy. The hydrodynamic scaling limit will be a corollary of what we shall call the relative entropy theorem. In Section 4, we give estimates based on entropy needed to prove the relative entropy theorem. Section 5, of most importance, deals with the proof of the relative entropy theorem by computing the relative entropy and its rate of change w.r.t. the local equilibrium states of our reference system. To perform this, we need a local ergodic theorem. Its proof is sketched in Section 6.

2. The model and its macroscopic equation

2.1. Presentation of the model

We look at the macroscopic scale, so we introduce the following rescaled space and time variables: $x_i = \varepsilon q_i$ and $t = \varepsilon^2 \tau$ in (1.1). This means that we set $x_i(t) = \varepsilon q_i(\varepsilon^{-2} t)$.

We keep the notation v_i for the macroscopic velocity of the ith particle (i.e. $v_i(t) \equiv v_i(\tau)$). We also keep the notation W_i for the Brownian motions terms because their laws are invariant under such rescaling of space and time.

We have then a system of N interacting particles evolving in $\mathbb{T}^d$, the d-dimensional unit torus. Particles have velocities in $\mathbb{R}^d$. We use the notation $(\underline{x}, \underline{v})$ for $\{(x_1, v_1), \dots, (x_N, v_N)\}$ which are, respectively, the positions and the velocities of the N individual particles. The system is described by the following stochastic differential equation in the phase space $(\mathbb{T}^d \times \mathbb{R}^d)^N$: for $i = 1, \dots, N$ and $\forall t \geqslant 0$:

$$\mathrm{d}x_i(t) = \varepsilon^{-1} v_i(t)\,\mathrm{d}t,$$

$$\mathrm{d}v_i(t) = -\varepsilon^{-2} \sum_{j \neq i} 2\nabla V(\varepsilon^{-1}(x_i(t) - x_j(t)))\,\mathrm{d}t - \frac{\varepsilon^{-2}}{2} \gamma v_i(t)\,\mathrm{d}t + \varepsilon^{-1} \sqrt{\frac{\gamma}{\beta}}\,\mathrm{d}W_i(t).$$

We make the following assumptions on V:

1. V is positive.
2. V is radial (i.e. V depends only on $|x|$), once continuously differentiable on $\mathbb{R}^d$ and it is assumed to be short range (i.e. $V(x)$ has compact support on $\mathbb{R}^d$: take $R > 0$ such that $V(x) = 0$ for $|x| \geqslant R$).

3. V is superstable: if Λ is a fixed bounded region of $\mathbb{R}^d$, there exist two constants $A > 0$ and $B \geqslant 0$ such that for all $n, x_1, \dots, x_n$,

$$\sum_{j \neq i}^{n} V(x_i - x_j) \geqslant -nB + \frac{1}{|\Lambda|} A n^2$$

(cf. Ruelle, 1970) where $|\Lambda|$ is the volume of Λ.

We shall assume that the number of particles depends on the scaling parameter $\varepsilon \in]0,1]$: typically, we will take $N = [\varepsilon^{-d}]$.

Consider the (macroscopic) Hamiltonian $\mathscr{H}_\varepsilon$ corresponding to the diffusion process $(\underline{x}(t), \underline{v}(t))$:

$$\mathscr{H}_\varepsilon(\underline{x}, \underline{v}) = \sum_{j=1}^{N} \frac{|v_j|^2}{2} + \sum_{(j \neq i)} V(\varepsilon^{-1}(x_j - x_i)).$$

The measure on $(\mathbb{T}^d \times \mathbb{R}^d)^N$ is given by

$$\mathrm{d}\mu_\varepsilon(\underline{x}, \underline{v}) = \frac{1}{Z_\varepsilon} \exp[-\beta \mathscr{H}_\varepsilon(\underline{x}, \underline{v})] d(\underline{x}, \underline{v})$$

where $\mathrm{d}(\underline{x}, \underline{v})$ denotes the Lebesgue measure on $(\mathbb{T}^d \times \mathbb{R}^d)^N$ and where Z_ε is the normalization constant making $\mathrm{d}\mu_\varepsilon$ a probability measure on $(\mathbb{T}^d \times \mathbb{R}^d)^N$.

The infinitesimal generator of the process can be written as a sum of a symmetric operator and an antisymmetric one with respect to μ_ε:

$$L_\varepsilon = S_\varepsilon + A_\varepsilon,$$

where

$$S_\varepsilon = \frac{\varepsilon^{-2}\gamma}{2} \sum_{i=1}^{N} \left(\frac{1}{\beta} \Delta_{v_i} - v_i \cdot \nabla_{v_i} \right),$$

$$A_\varepsilon = \varepsilon^{-1} \sum_{i=1}^{N} \left(v_i \cdot \nabla_{x_i} - \varepsilon^{-1} \sum_{j \neq i} 2 \nabla V(\varepsilon^{-1}(x_i - x_j)) \cdot \nabla_{v_i} \right).$$

Here, ∇_{v_i} and Δ_{v_i} stand, respectively, for the gradient w.r.t. v_i and the Laplacian w.r.t. v_i and ∇_{x_i} stands for the gradient w.r.t. x_i.

We start at $x(0)$ with an initial distribution of diffusion which has a density denoted $f_\varepsilon^0(\underline{x}, \underline{v})$ with respect to the measure μ_ε. The density at time t with respect to μ_ε, denoted by $f_\varepsilon^t(\underline{x}, \underline{v})$, is given as a solution of the forward equation:

$$\frac{\partial}{\partial t} f_\varepsilon^t = L_\varepsilon^* f_\varepsilon^t,$$

$$f_\varepsilon^t|_{t=0} = f_\varepsilon^0, \tag{2.1}$$

where L_ε^* is the adjoint of L_ε w.r.t. μ_ε.

2.2. Thermodynamic functions

To present our results, we introduce some thermodynamic functions associated to our system of particles interacting via the potential V.

Taking Λ to be some regular region of $\mathbb{R}^d$, we start with our Hamiltonian in the microscopic scale on configurations of n points in the phase space $(\Lambda \times \mathbb{R}^d)^n$:

$$\mathscr{H}_n(\underline{q}, \underline{v}) = \sum_{j=1}^{n} \frac{|v_j|^2}{2} + \sum_{(j \neq i)} V(q_i - q_j).$$

We can define the grand canonical partition function as

$$Z_\Lambda(\lambda, \beta) = \sum_{n=0}^{+\infty} \frac{\mathrm{e}^{\lambda\beta n}}{n!} \int_{\Lambda^n} \mathrm{d}\underline{q} \int_{\mathbb{R}^{dn}} \mathrm{d}\underline{v} \exp[-\beta \mathscr{H}_n(\underline{q}, \underline{v})],$$

where $\lambda \in \mathbb{R}$ is called the chemical potential.

The thermodynamic pressure is defined by the following limit:

$$\tilde{P}(\lambda, \beta) = \lim_{\Lambda \to \mathbb{R}^d} \frac{1}{|\Lambda|\beta} \log Z_\Lambda(\lambda, \beta)$$

This limit exists and defines a convex and continuous function of λ and β (see Ruelle, 1969, 1970).

For a fixed inverse of temperature $\beta > 0$, the Gibbs measure corresponding to λ on $\mathbb{R}^d \times \mathbb{R}^d$ is invariant under space translations and has the DLR property, namely the conditional distribution of the possible configurations in a box Λ, given the outside configuration $\{(q', v') | q'_j \in \Lambda^c\}$, is given by

$$\mu(\mathrm{d}q_1, \ldots, \mathrm{d}q_n, \mathrm{d}v_1, \ldots, \mathrm{d}v_n \,|\, (q', v'), q'_j \in \Lambda^c)$$

$$= \frac{1}{Z} \frac{\mathrm{e}^{\lambda\beta n}}{n!} \exp\left[-\beta\left(\mathscr{H}_n + \sum_{i=1}^{n} \sum_{j=1}^{\infty} V(q_i - q'_j)\right)\right] \mathrm{d}q_1 \ldots \mathrm{d}q_n \mathrm{d}v_1 \ldots \mathrm{d}v_n$$

on $\bigcup_n (\Lambda \times \mathbb{R}^d)^n$ for $n = 1, 2, \ldots$. The term $\sum_{i=1}^{n} \sum_{j=1}^{\infty} V(q_i - q'_j)$ describes the interaction energy due to the configuration outside the box Λ (see Spohn (1991) or Olla et al. (1993)) and Z (which depends on the outside configuration) is a normalization constant.

Then, given assumptions (1)–(3) on V (see Section 2.1), the general theory of equilibrium statistical mechanics provides a non-empty set $U \subseteq \mathbb{R}$ of possible values of λ such that U is an open set of $\mathbb{R}$ and for any $\lambda \in U$, there exists an unique infinite volume Gibbs state. This Gibbs measure is ergodic w.r.t. space translations. The average density of particles ρ is given as smooth function of $\lambda \in U$ by

$$U \to W \doteq \partial_\lambda \tilde{P}(U, \beta) \subseteq \mathbb{R}^+,$$

$$\lambda \mapsto \rho(\lambda) = \partial_\lambda \tilde{P}(\lambda, \beta).$$

Of course, this function can be inverted to yield $\lambda = \lambda(\rho)$ as a function of ρ. For that, we introduce the Helmholtz free energy $a(\rho, \beta)$ as the Legendre transform of the pressure

$$a(\rho, \beta) = \sup_{\lambda \in \mathbb{R}} \{\lambda\rho - \tilde{P}(\lambda, \beta)\}$$

and now we can also see the chemical potential as a function of the density by the smooth 1–1 map

$$W \to U,$$

$$\rho \mapsto \lambda(\rho) = \partial_\rho a(\rho, \beta). \tag{2.2}$$

Therefore, we can consider the pressure P as function of the density (in the one phase region, the pressure is a smooth and strictly increasing function of the density): set $P(\rho) = \tilde{P}(\lambda(\rho))$.

Consider then the non-linear differential equation:

$$\partial_t \rho(t,x) = \frac{2}{\gamma}\, \Delta P(\rho(t,x)). \tag{2.3}$$

Let $K \subseteq W$ be a compact set such that $\mathrm{d}(K, \mathbb{R} \setminus W) > \eta$ for some $\eta > 0$.

If we give an initial data $\rho_0 \in \mathscr{C}^\infty(\mathbb{T}^d)$ to our Eq. (2.3) such that $\rho_0(\mathbb{T}^d) \subset K$, then, by the maximum principle, a $\mathscr{C}^\infty$-solution $\rho(t,x)$ exists and verifies $\rho(t,x) \in K$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{T}^d$.

Clearly, by map (2.2), the corresponding λ is a $\mathscr{C}^\infty$ function $\lambda(t,x)$. It lies in the compact $\tilde{K} = \partial_\rho a(K, \beta)$ with $d(\tilde{K}, \mathbb{R} \setminus U) > \tilde{\eta}$ for some $\tilde{\eta} > 0$. This implies that λ and its derivates are bounded for all $(t,x) \in \mathbb{R}^+ \times \mathbb{T}^d$.

Let us introduce the local Gibbs measure associated to the macroscopic evolution $\rho(t,x)$ as the probability measure on $(\mathbb{T}^d \times \mathbb{R}^d)^N$ which is absolutely continuous w.r.t. $\mu_\varepsilon^t(\underline{x}, \underline{v})$ with density:

$$\psi_\varepsilon^t(\underline{x}, \underline{v}) = \frac{1}{C_\varepsilon(t)} \exp\left[\beta \sum_{i=1}^N \lambda(t, x_i) - 2\varepsilon\, \frac{\beta}{\gamma} \sum_{i=1}^N (\nabla\lambda)(t, x_i) \cdot v_i\right], \tag{2.4}$$

where $C_\varepsilon(t)$ is the normalization constant.

Now, we expose our main result:

We shall assume that the initial distribution of the process satisfies the following (entropy) limit:

$$\lim_{\varepsilon \to 0} \varepsilon^d \int_{(\mathbb{T}^d \times \mathbb{R}^d)^N} \mathrm{d}\mu(\underline{x}, \underline{v}) f_\varepsilon^0(\underline{x}, \underline{v}) \log \frac{f_\varepsilon^0(\underline{x}, \underline{v})}{\psi_\varepsilon^0(\underline{x}, \underline{v})} = 0. \tag{2.5}$$

Then, consider a C^∞ function $J : \mathbb{T}^d \to \mathbb{R}^d$.

For any $t \in \mathbb{R}^+$, we define $A_{N,J,\delta}^t$ the set of configurations $(x_1, \ldots, x_N)$ such that

$$A_{N,J,\delta}^t = \left\{ \underline{x} \in \mathbb{T}^{dN} : \left| \frac{1}{N} \sum_{i=1}^N J(x_i) - \int_{\mathbb{T}^d} \mathrm{d}x\, J(x) \rho(t,x) \right| > \delta \right\}$$

with $\rho(t,x)$ is the solution of (2.3).

The aim of this paper is to prove the following theorem.

Theorem 2.1 (Hydrodynamic Scaling Limit). *Let $t \in \mathbb{R}^+$. Let $f_\varepsilon^t(\underline{x}, \underline{v})$ be the solution of Eq.* (2.1) *with f_ε^0 satisfying condition* (2.5). *Then, for every $\mathscr{C}^\infty$ function $J : \mathbb{T}^d \to \mathbb{R}^d$ and for every $\delta > 0$*

$$\lim_{\varepsilon \to 0} \int_{A_{N,J,\delta}^t} f_\varepsilon^t(\underline{x}, \underline{v})\, \mathrm{d}\mu_\varepsilon(\underline{x}, \underline{v}) = 0.$$

3. The relative entropy method

To prove Theorem 2.1, we use the relative entropy method introduced by Yau (1991). Firstly, let us recall some definitions and properties.

For μ and ν two measures of probability on the same measurable space Ω, the well-known entropy inequality states for any measurable function F in $L^1(\mathrm{d}\nu)$ and for all $\kappa > 0$

$$\int_\Omega F\,\mathrm{d}\mu \leqslant \frac{1}{\kappa}H(\mu|\nu) + \frac{1}{\kappa}\log\int_\Omega \exp(\kappa F)\,\mathrm{d}\nu, \tag{3.1}$$

$H(\mu|\nu)$ is called the relative entropy of μ with respect to ν.

Moreover, if $\mu \ll \nu$, we have:

$$H(\mu|\nu) = \int_\Omega \log\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\,\mathrm{d}\mu.$$

As a consequence of (3.1), for any measurable set A, we have that

$$\mu(A) \leqslant \frac{\log 2 + H(\mu\,|\,\nu)}{\log(1 + 1/\nu(A))}. \tag{3.2}$$

For every $t \in \mathbb{R}^+$, we define the following functional

$$H_\varepsilon(t) \doteq H(f_\varepsilon^t\,\mathrm{d}\mu_\varepsilon|\psi_\varepsilon^t\,\mathrm{d}\mu_\varepsilon) = \int_{(\mathbb{T}^d\times\mathbb{R}^d)^N} \mathrm{d}\mu_\varepsilon(\underline{x},\underline{v}) f_\varepsilon^t(\underline{x},\underline{v}) \log\frac{f_\varepsilon^t(\underline{x},\underline{v})}{\psi_\varepsilon^t(\underline{x},\underline{v})}.$$

Note that (2.5) says that $\lim_{\varepsilon\to 0}\varepsilon^d H_\varepsilon(0) = 0$.

Let us now give our main result.

Theorem 3.1 (Relative Entropy Theorem). *Let $f_\varepsilon^t(\underline{x},\underline{v})$ be the solution of* (2.1) *with f_ε^0 satisfying condition* (2.5). *Then, for any $t \in \mathbb{R}^+$:*

$$\lim_{\varepsilon\to 0}\varepsilon^d H_\varepsilon(t) = 0.$$

Theorem 2.1 becomes a corollary of Theorem 3.1. Indeed, using the large deviation theory for the local Gibbs state (2.4) (see Corollary 5.8 of Olla et al. (1993)), we get the existence of a constant $C(\varepsilon, J) > 0$ such that

$$E_{\psi_\varepsilon^t}[1_{A_{N,J,\delta}^t}] \leqslant \exp[-C(\varepsilon,J)N],$$

where $E_f[\,\cdot\,]$ denotes the expectation w.r.t. the measure $f\,\mathrm{d}\mu_\varepsilon$ and 1_Γ the characteristic function of the set Γ.

On the other hand, from the entropy estimate (3.2), we get

$$E_{f_\varepsilon^t}[1_{A_{N,J,\delta}}] \leqslant \frac{\log 2 + H_\varepsilon(t)}{\log(1 + E_{\psi_\varepsilon^t}[1_{A_{N,J,\delta}}]^{-1})}$$

so that there exists some constant $C > 0$ such that,

$$E_{f_\varepsilon^t}[1_{A_{N,J,\delta}}] \leqslant C\left(\frac{1}{N} + \frac{1}{N}H_\varepsilon(t)\right).$$

Since $N = [\varepsilon^{-d}]$, the limit of the above expectation goes to 0 as $\varepsilon \to 0$.

4. Some estimates based on entropy

To simplify notations, we take, only in this section, $\gamma = 1$ and $\beta = 1$.

In order to prove Theorem 3.1, we need some estimates based on entropy and the Dirichlet form.

We denote by $H(f_\varepsilon^t|1)$ the relative entropy of $f_\varepsilon^t \mathrm{d}\mu_\varepsilon$ w.r.t. $\mathrm{d}\mu_\varepsilon$. We introduce the Dirichlet form defined as

$$I_\varepsilon(f) = \frac{1}{2}\int_{(\mathbb{T}^d\times\mathbb{R}^d)^N} \mathrm{d}\mu_\varepsilon(\underline{x},\underline{v}) \sum_{i=1}^N \frac{|\nabla_{v_i} f(\underline{x},\underline{v})|^2}{f(\underline{x},\underline{v})}. \tag{4.1}$$

Proposition 4.1. (i) $\forall t \geqslant 0$, *there exists a constant* C *such that*

$$H(f_\varepsilon^t|1) \leqslant C\varepsilon^{-d},$$

(ii) $\forall t \geqslant 0$, $\int_0^t I_\varepsilon(f_\varepsilon^s)\,\mathrm{d}s \leqslant C\varepsilon^{2-d}$.

Proof. Because f_ε^t is solution of (2.1), we get by an explicit computation

$$\frac{\mathrm{d}H(f_\varepsilon^t|1)}{\mathrm{d}t} = -\frac{\varepsilon^{-2}}{2}\int \mathrm{d}\mu_\varepsilon(\underline{x},\underline{v}) \sum_{i=1}^N \frac{|\nabla_{v_i} f_\varepsilon^t(\underline{x},\underline{v})|^2}{f_\varepsilon^t(\underline{x},\underline{v})} = -\varepsilon^{-2} I_\varepsilon(f_\varepsilon^t) \tag{4.2}$$

so, $\mathrm{d}H(f_\varepsilon^t|1)/\mathrm{d}t \leqslant 0$.

Then the entropy decreases in t so $H(f_\varepsilon^t|1) \leqslant H(f_\varepsilon^0|1)$.

But

$$\begin{aligned}
H(f_\varepsilon^0|1) &= H_\varepsilon(0) - \log C_\varepsilon(0) + \int \mathrm{d}\mu_\varepsilon f_\varepsilon^0 \sum_i (\lambda(0,x_i) - 2\varepsilon\nabla\lambda(0,x_i)\cdot v_i(0)) \\
&\leqslant H_\varepsilon(0) - \int \mathrm{d}\mu_\varepsilon \sum_{i=1}^N (\lambda(0,x_i) - 2\varepsilon\nabla\lambda(0,x_i)\cdot v_i(0)) \\
&\quad + \int \mathrm{d}\mu_\varepsilon f_\varepsilon^0 \sum_{i=1}^N (\lambda(0,x_i) - 2\varepsilon\nabla\lambda(0,x_i)\cdot v_i(0)) \\
&\leqslant H_\varepsilon(0) + 2\|\lambda(0,\cdot)\|_\infty N \\
&\quad + \int \mathrm{d}\mu_\varepsilon \sum_{i=1}^N (2\varepsilon|\nabla\lambda(0,x_i)||v_i(0)|) + \int \mathrm{d}\mu_\varepsilon f_\varepsilon^0 \sum_{i=1}^N (2\varepsilon|\nabla\lambda(0,x_i)||v_i(0)|)
\end{aligned}$$

by using Jensen inequality for the first inequality. Then, there exists a constant $C_1 = 2\|\lambda(0,\cdot)\|_\infty + C$ (where C is the constant coming from (2.5)) such that

$$H(f_\varepsilon^0|1) \leqslant C_1\varepsilon^{-d} + \int \mathrm{d}\mu_\varepsilon \sum_{i=1}^N (2\varepsilon|\nabla\lambda(0,x_i)||v_i(0)|)$$

$$+ \int \mathrm{d}\mu_\varepsilon f_\varepsilon^0 \sum_{i=1}^N (2\varepsilon|\nabla\lambda(0,x_i)||v_i(0)|). \tag{4.3}$$

Concerning the second term of the r.h.s. of inequality (4.3), we get

$$\int \mathrm{d}\mu_\varepsilon \sum_{i=1}^N (2\varepsilon|\nabla\lambda(0,x_i)||v_i(0)|) \leqslant 2\varepsilon N\|\nabla\lambda(0,\cdot)\|_\infty \int \mathrm{d}\mu_\varepsilon|v_1(0)| \leqslant C_2\varepsilon^{-d+1}, \tag{4.4}$$

where $C_2 = 2\|\nabla\lambda(0,\cdot)\|_\infty E_{\mu_\varepsilon}(|v_1(0)|)$ which is finite because v_1 is gaussian.

For the third term of the r.h.s. of (4.3), we get by Schwarz inequality:

$$\int \mathrm{d}\mu_\varepsilon f_\varepsilon^0 \sum_{i=1}^N (2\varepsilon|\nabla\lambda(0,x_i)||v_i(0)|) \leqslant \varepsilon\tilde{C}_2 N^{1/2}\left(\int \mathrm{d}\mu_\varepsilon f_\varepsilon^0 \sum_{i=1}^N v_i^2\right)^{1/2}$$

$$\leqslant \varepsilon\tilde{C}_2\varepsilon^{-d/2}(4H(f_\varepsilon^0|1) + 4\log(\sqrt{2}^{dN}))^{1/2} \tag{4.5}$$

where we used the entropic inequality (3.1) with $\kappa = 1/4$ and with $\tilde{C}_2 = 2\|\nabla\lambda(0,\cdot)\|_\infty$.

Then, inserting (4.4) and (4.5) in (4.3), we finally get that

$$H(f_\varepsilon^0|1) \leqslant C_3\varepsilon^{-d} + \tilde{C}_2\varepsilon^{-d/2}\sqrt{4H(f_\varepsilon^0|1) + C_4\varepsilon^{-d}},$$

where $C_3 = C_1 + \varepsilon C_2$ and $C_4 = 4d\log\sqrt{2}$.

Studying the inequality $x \leqslant c_1\varepsilon^{-d} + c_2\varepsilon^{-d/2}\sqrt{4x + c_3\varepsilon^{-d}}$, it is easy to see that $x \leqslant C_5\varepsilon^{-d}$ (by contradiction for instance it is immediate), we can conclude that there exists a constant C_5 such that $H(f_\varepsilon^0|1) \leqslant C_5\varepsilon^{-d}$. (i) of Lemma 4.1 follows.

From (4.2), we easily get (ii) of Lemma 4.1 by integration. □

A corollary of this entropic bound is the following theorem:

Theorem 4.2. *Let ϕ be a continuous function on $\mathbb{R}^d$ with compact support. Then, there exists a constant $C_\phi > 0$ such that for any $\varepsilon \in (0,1]$ and $t \in (0,+\infty[$*

$$E_{f_\varepsilon^t}\left[\varepsilon^d \sum_{i\neq j} \phi(\varepsilon^{-1}(x_i - x_j))\right] \leqslant C_\phi.$$

Proof. By a straightforward extension to higher dimensions of Lemma 4.2 of Varadhan (1991), it is sufficient to prove Theorem 4.2 replacing ϕ by the potential V. Then, the proof is a straightforward extension to higher dimensions of Lemma 4.1 of Varadhan (1991) which uses Proposition 4.1. □

Now, we give an ultimate result we shall need in the next section.

Lemma 4.3. *For any* $t \geqslant 0$, *we have that*

$$\lim_{\varepsilon\to 0}\int_0^t \mathrm{d}s\, E_{f_\varepsilon^s}\left[\varepsilon^d\sum_{i=1}^N(|v_i|^2-1)\Delta\lambda(s,x_i)\right]=0.$$

Proof. Firstly, integrating by parts and then using Schwarz inequality, we compute that

$$\begin{aligned}&\int_0^t \mathrm{d}sE_{f_\varepsilon^s}\left[\varepsilon^d\sum_{i=1}^N(|v_i|^2-1)\Delta\lambda(s,x_i)\right]\\
&\quad=\int_0^t \mathrm{d}sE_{\mu_\varepsilon}\left[\varepsilon^d\sum_{i=1}^N v_i\cdot\Delta\lambda(s,x_i)\cdot\partial_{v_i}f_\varepsilon^s(\underline{x},\underline{v})\right]\\
&\quad=\sqrt{2}\left(\int_0^t \mathrm{d}sE_{f_\varepsilon^s}\left[\varepsilon^d\sum_{i=1}^N|v_i|^2\cdot\Delta\lambda(s,x_i)^2\right]\right)^{1/2}\left(\varepsilon^d\int_0^t\mathrm{d}sI_\varepsilon(f_\varepsilon^s)\right)^{1/2}\end{aligned}$$

By Proposition 4.1(ii),

$$\varepsilon^d\int_0^t\mathrm{d}sI_\varepsilon(f_\varepsilon^s)\leqslant 2C\varepsilon^2. \tag{4.6}$$

Moreover, using the entropic inequality (3.1) with $\kappa=\frac{1}{4}\varepsilon^{-d}$, we get that

$$\begin{aligned}E_{f_\varepsilon^s}\left[\varepsilon^d\sum_{i=1}^N|v_i|^2\cdot\Delta\lambda(s,x_i)^2\right]&\leqslant\|\Delta\lambda^2\|_\infty\left\{4C+4\varepsilon^d\log\int\exp^{1/4}\sum_{i=1}^N|v_i|^2\,\mathrm{d}\mu_\varepsilon\right\}\\
&\leqslant\|\Delta\lambda^2\|_\infty\{4C+4\varepsilon^dNd\log(\sqrt{2})\}\end{aligned}$$

Therefore, for all $t \geqslant 0$, we get that $\int_0^t \mathrm{d}sE_{f_\varepsilon^s}[\varepsilon^d\sum_{i=1}^N|v_i|^2\Delta\lambda(s,x_i)^2]$ is finite. The result follows. □

5. Proof of Theorem 3.1

To prove Theorem 3.1, we make several steps. The first one consists on an estimation of $\varepsilon^d H_\varepsilon(t)$. In the second step, we rewrite quantities obtained in term of local empirical quantities in order to apply an ergodic theorem in the step 3. We will conclude on step 4.

Step 1: It consists on an estimation of $\mathrm{d}H_\varepsilon(t)/\mathrm{d}t$ using Lemma 1.4 of Chapter 6 in Kipnis and Landim (1999) which gives an upper bound for the entropy production. The aim of this part is to arrive to the following bound:

$$\lim_{\varepsilon\to 0}\int_0^t \left\{\varepsilon^d\,\frac{\mathrm{d}H_\varepsilon(s)}{\mathrm{d}s} - E_{f_\varepsilon^s}\left[\frac{\beta}{\gamma}\phi_\varepsilon(s,\underline{x},\underline{v},\lambda)\right] + \beta\int_{\mathbb{T}^d}\mathrm{d}z\,\dot{\lambda}(s,z)\rho(s,z)\right\}\mathrm{d}s \leqslant 0,$$

where ϕ_ε is a function we shall define later.

Lemma 1.4 of Chapter 6 in Kipnis and Landim (1999) says:

$$\varepsilon^d\,\frac{\mathrm{d}H_\varepsilon(t)}{\mathrm{d}t} \leqslant \varepsilon^d\int \mathrm{d}\mu_\varepsilon(\underline{x},\underline{v})f_\varepsilon^t(\underline{x},\underline{v})\left(\frac{L_\varepsilon^*\psi_\varepsilon^t(\underline{x},\underline{v})}{\psi_\varepsilon^t(\underline{x},\underline{v})} - \partial_t\log\psi_\varepsilon^t(\underline{x},\underline{v})\right). \tag{5.1}$$

Using the fact that: $L_\varepsilon^* = S_\varepsilon - A_\varepsilon$, we make several computations, separately.
(i)

$$\frac{L_\varepsilon^*\psi_\varepsilon^t(\underline{x},\underline{v})}{\psi_\varepsilon^t(\underline{x},\underline{v})} = \frac{\beta}{\gamma}\left(2\sum_{i=1}^N|\nabla\lambda(t,x_i)|^2 + 2\sum_{i=1}^N(|v_i|^2-1)\cdot\Delta\lambda(t,x_i)\right.$$
$$\left. -\ 4\varepsilon^{-1}\sum_{j\neq i}\nabla V(\varepsilon^{-1}(x_i-x_j))\cdot\nabla\lambda(t,x_i) + 2\sum_{i=1}^N\Delta\lambda(t,x_i)\right). \tag{5.2}$$

Concerning the third term of the r.h.s. of (5.2), by skew-symmetry of ∇V, we have:

$$-\varepsilon^{-1}\sum_{j\neq i}\nabla V(\varepsilon^{-1}(x_i(t)-x_j(t)))\cdot\nabla\lambda(t,x_i)$$
$$= -\frac{\varepsilon^{-1}}{2}\sum_{j\neq i}\nabla V(\varepsilon^{-1}(x_i(t)-x_j(t)))\cdot[\nabla\lambda(t,x_i)-\nabla\lambda(t,x_j)]$$
$$=\frac{1}{2}\sum_{j\neq i}\sum_{\alpha,\sigma=1}^d\psi_{\alpha\sigma}(\varepsilon^{-1}(x_i-x_j))\partial_{\alpha\sigma}\lambda(t,x_i) + R_V(\underline{x}) \tag{5.3}$$

with $\psi_{\alpha\sigma}(q) = -q^\sigma\partial_\alpha V(q)$ where q^σ is the σth component of $q\in\mathbb{T}^d$ and $\partial_{\alpha\sigma} = \partial^2/\partial q^\alpha\partial q^\sigma$. $R_V(\underline{x})$ is explicit and its expression is

$$R_V(\underline{x}) = -\frac{\varepsilon^{-1}}{2}\sum_{j\neq i}\sum_{\alpha,\sigma}[\nabla\lambda(t,x_i)-\nabla\lambda(t,x_j)$$
$$-(x_i-x_j)^\sigma\partial_{\alpha\sigma}\lambda(t,x_i)]\partial_\alpha V(\varepsilon^{-1}(x_i-x_j)).$$

Since ∇V has a compact support, by a Taylor's expansion, we can estimate that

$$|R_V(\underline{x})| \leqslant \mathrm{o}(\varepsilon)\sum_{j\neq i}|\nabla V|(\varepsilon^{-1}(x_i-x_j)). \tag{5.4}$$

Then, we insert (5.3) and (5.4) in (5.2) to obtain

$$\frac{L_\varepsilon^* \psi_\varepsilon^t(\underline{x},\underline{v})}{\psi_\varepsilon^t(\underline{x},\underline{v})} \leqslant 2\frac{\beta}{\gamma}\left(\sum_{i=1}^N |\nabla\lambda(t,x_i)|^2 + \sum_{i=1}^N (|v_i|^2-1)\cdot\Delta\lambda(t,x_i)\right.$$
$$\left. + \sum_{i=1}^N \Delta\lambda(t,x_i) + \sum_{j\neq i}\sum_{\alpha,\sigma}\psi_{\alpha\sigma}(\varepsilon^{-1}(x_i-x_j))\partial_{\alpha\sigma}\lambda(t,x_i) + R_V(\underline{x})\right). \tag{5.5}$$

(ii). More, we compute $\partial_t \log\psi_\varepsilon^t(\underline{x},\underline{v})$:

$$\partial_t \log\psi_\varepsilon^t(\underline{x},\underline{v}) = -\frac{\dot{C}_\varepsilon(t)}{C_\varepsilon(t)} + \beta\sum_{i=1}^N \dot{\lambda}(t,x_i) - 2\varepsilon\frac{\beta}{\gamma}\sum_{i=1}^N \nabla\dot{\lambda}(t,x_i)\cdot v_i$$

where $\dot{f}(t,x) \doteq \partial f(t,x)/\partial t$.

It means that

$$\partial_t \log\psi_\varepsilon^t(\underline{x},\underline{v}) = -E_{\psi_\varepsilon^t}\left[\beta\sum_{i=1}^N \dot{\lambda}(t,x_i) - 2\varepsilon\frac{\beta}{\gamma}\sum_{i=1}^N \nabla\dot{\lambda}(t,x_i)\cdot v_i\right]$$
$$+\beta\sum_{i=1}^N \dot{\lambda}(t,x_i) - 2\varepsilon\frac{\beta}{\gamma}\sum_{i=1}^N \nabla\dot{\lambda}(t,x_i)\cdot v_i. \tag{5.6}$$

(iii) Now, inserting (5.5) and (5.6) into (5.1), using (5.4), we give a final estimation of $\varepsilon^d \mathrm{d}H_\varepsilon(t)/\mathrm{d}t$

$$\varepsilon^d\frac{\mathrm{d}H_\varepsilon(t)}{\mathrm{d}t} \leqslant E_{f_\varepsilon^t}\left[\frac{\beta}{\gamma}\phi_\varepsilon(t,\underline{x},\underline{v},\lambda)\right] + \varepsilon^d E_{\psi_\varepsilon^t}\left[\beta\sum_{i=1}^N \dot{\lambda}(t,x_i)\right]$$
$$+\Xi_{1,\varepsilon}(t) + \Xi_{2,\varepsilon}(t) + \Xi_{3,\varepsilon}(t) + \Xi_{4,\varepsilon}(t) \tag{5.7}$$

where

$$\phi_\varepsilon(t,\underline{x},\underline{v},\lambda) = 2\varepsilon^d\sum_{i=1}^N |\nabla\lambda(t,x_i)|^2 + 2\varepsilon^d\sum_{i=1}^N \Delta\lambda(t,x_i)$$
$$+2\varepsilon^d\sum_{i,j}\sum_{\alpha,\sigma}\partial_{\alpha\sigma}\lambda(t,x_i)\psi_{\alpha\sigma}(\varepsilon^{-1}(x_i-x_j)) - \varepsilon^d\gamma\sum_{i=1}^N \dot{\lambda}(t,x_i) \tag{5.8}$$

and

$$\Xi_{1,\varepsilon}(t) \leqslant \varepsilon^d\frac{\beta}{\gamma}\mathrm{o}(\varepsilon)E_{f_\varepsilon^t}\left[\sum_{i,j}|\nabla V|(\varepsilon^{-1}(x_i-x_j))\right], \tag{5.9}$$

$$\Xi_{2,\varepsilon}(t) = 2\,\frac{\beta}{\gamma}\,\varepsilon^{d+1} E_{\psi_\varepsilon^t}\left(\sum_{i=1}^{N} \nabla \dot{\lambda}(t,x_i)\cdot v_i\right),$$

$$\Xi_{3,\varepsilon}(t) = 2\,\frac{\beta}{\gamma}\,\varepsilon^{d+1} E_{f_\varepsilon^t}\left[\sum_{i=1}^{N} \nabla \dot{\lambda}(t,x_i)\cdot v_i\right], \tag{5.10}$$

$$\Xi_{4,\varepsilon}(t) = 2\,\frac{\beta}{\gamma}\,\varepsilon^{d} E_{f_\varepsilon^t}\left[\sum_{i=1}^{N} (|v_i|^2-1)\cdot \Delta\lambda(t,x_i)\right]. \tag{5.11}$$

By Theorem 4.2, we get that $\lim_{\varepsilon\to 0}\sup_{t\geqslant 0}|\Xi_{1,\varepsilon}(t)| = 0$. It is also clear that $\lim_{\varepsilon\to 0}\sup_{t\geqslant 0}|\Xi_{3,\varepsilon}(t)| = 0$.

Moreover, by an explicit computation for the local Gibbs state ψ_ε^t, using the Large Deviation Principle (see Section 5 of Butta and Lebowitz (1999)), we know that for any $t \geqslant 0$

$$\lim_{\varepsilon\to 0} \varepsilon^d E_{\psi_\varepsilon^t}\left(\beta\sum_{i=1}^{N} \dot{\lambda}(t,x_i)\right) = \beta\int_{\mathbb{T}^d} \mathrm{d}z\,\dot{\lambda}(t,z)\rho(t,z) \tag{5.12}$$

and

$$\limsup_{\varepsilon\to 0}\sup_{t\geqslant 0} \Xi_{2,\varepsilon}(t) = \limsup_{\varepsilon\to 0}\sup_{t\geqslant 0} \varepsilon^{d+1} E_{\psi_\varepsilon^t}\left(\sum_{i} \nabla\dot{\lambda}(t,x_i)\cdot v_i\right) = 0. \tag{5.13}$$

Finally, collecting together (5.7), (5.9), (5.10),(5.12) and using Lemma 4.3 to $\Xi_{4,\varepsilon}(t)$, we get that for any $t \geqslant 0$

$$\limsup_{\varepsilon\to 0} \int_0^t \left\{ \varepsilon^d\,\frac{\mathrm{d}H_\varepsilon(s)}{\mathrm{d}s} - E_{f_\varepsilon^s}\left[\frac{\beta}{\gamma}\phi_\varepsilon(s,\underline{x},\underline{v},\lambda)\right] - \beta\int_{\mathbb{T}^d} \mathrm{d}z\,\dot{\lambda}(s,z)\rho(s,z)\right\}\mathrm{d}s \leqslant 0. \tag{5.14}$$

Step 2: The aim of this step is to write $\phi_\varepsilon(s,\underline{x},\underline{v},\lambda)$ in term of local empirical quantities.

Let Ω be the space of particles configurations on $\mathbb{R}^d$: Ω is a subset of $\mathbb{R}^d$ which is locally finite. Let $\omega\in\Omega$. Given $z\in\mathbb{T}^d$, for any $\underline{x}\in\mathbb{T}^{dN}$, we construct a configuration $\omega_\varepsilon^z\in\Omega$ by setting:

$$\omega_\varepsilon^z := \{\varepsilon^{-1}(x_i - z)\,|\,|x_i - z| < \tfrac{1}{4}\}.$$

Clearly, it is well defined in every compact set inside the cube of $\mathbb{R}^d$ of side $1/2\varepsilon$ and centered at the origin.

Consider then a function $F(\omega)$ on the configuration of points in $\mathbb{R}^d$ which is bounded, continuous and localized in some finite d-interval $[-l,l]^d$. If ε is small enough, $F(\omega_\varepsilon^z)$ is well defined.

For $k\in\mathbb{N}$, let D_k be the d-dimensional cube

$$D_k=\{\ y\in\mathbb{R}^d, |y_i|\leqslant k,\quad i=1\dots d\}.$$

For any local function F, we define F_k its spatial average over D_k:

$$F_k(w)=\frac{1}{|D_k|}\int_{D_k}\mathrm{d}q\,F(\tau_q w),$$

where τ_q stands for the space translation by q.

Let h be a non-negative function on $\mathbb{R}^d$ with compact support and such that $\int_{\mathbb{R}^d}h(z)\,\mathrm{d}z=1$.

We define the following local function on Ω

$$\Theta(\omega)=\sum_{q_i\in\omega}h(q_i)$$

and set $\Theta_k(\omega)$ its average over D_k defined as for $F_k(\omega)$.

We also introduce for α, $\sigma=1\dots d$

$$G^{\alpha\sigma}(w)=\sum_{q,q'\in w,q\neq q'}h(q)\psi_{\alpha\sigma}(q-q')$$

and let $G_k^{\alpha\sigma}$ be its average over D_k.

By Lemma 3.2 of Butta and Lebowitz (1999), we have the two following results: $\forall J\in C^0(\mathbb{T}^d)$

$$\limsup_{k\to+\infty}\limsup_{\varepsilon\to 0}\sup_{t\geqslant 0}E_{f_\varepsilon^t}\left|\varepsilon^d\sum_{i=1}^N J(x_i)-\int_{\mathbb{T}^d}\mathrm{d}z\,J(z)\Theta_k(\omega_\varepsilon^z)\right|=0 \tag{5.15}$$

and

$$\limsup_{k\to+\infty}\limsup_{\varepsilon\to 0}\sup_{t\geqslant 0}E_{f_\varepsilon^t}\left|\varepsilon^d\sum_{i\neq j}^N\sum_{\alpha,\sigma}^N\psi_{\alpha\sigma}(\varepsilon^{-1}(x_i-x_j))J(x_i)-\int_{\mathbb{T}^d}\mathrm{d}z\,J(z)G_k^{\alpha\sigma}(\omega_\varepsilon^z)\right|=0. \tag{5.16}$$

Conclusion of the step 2: Finally, this permits us to replace $\phi_\varepsilon(s,\underline{x},\underline{v},\lambda)$ (given by (5.8)) by $\int_{\mathbb{T}^d}J_k(s,z)\,\mathrm{d}z$ where

$$J_k(s,z)=\left[\dot\lambda(s,z)-\frac{2}{\gamma}|\nabla\lambda(s,z)|^2-\frac{2}{\gamma}\,\Delta\lambda(s,z)\right]\Theta_k(\omega_\varepsilon^z)$$
$$-\frac{2}{\gamma}\sum_{\alpha,\sigma}\partial_{\alpha\sigma}\lambda(s,z)G_k^{\alpha\sigma}(\omega_\varepsilon^z).$$

And finally get that

$$\limsup_{k\to+\infty}\limsup_{\varepsilon\to 0}\int_0^t\left\{\varepsilon^d\frac{\mathrm{d}H_\varepsilon(s)}{\mathrm{d}s}+E_{f_\varepsilon^s}\right.$$
$$\left.\left(\beta\int_{\mathbb{T}^d}\mathrm{d}z\,J_k(s,z)-\dot\lambda(t,z)\rho(s,z)\right)\right\}\mathrm{d}s\leqslant 0. \tag{5.17}$$

Step 3: We want to substitute the spatial average $G_k^{\alpha\sigma}$ with a function of the empirical density Θ_k.

In this part, we introduce some cutoffs because the empirical densities $\Theta_k(\omega)$ characterizing the local equilibrium states may not lie in the admissible region W defined by (2.2).

Let $\mathscr{C}$ be a compact set such that $K \subset \mathscr{C} \subset W$ and $d(\mathscr{C}, \mathbb{R}^+ \setminus W) \geqslant \eta/2$ and $\mathrm{d}(K, \mathbb{R}^+ \setminus \mathscr{C}) \geqslant \eta/2$ (η is defined in Section 2.2).

We define the following local function:

$$\sigma_k(w) = 1_{\mathscr{C}}(\Theta_k(w)).$$

We also denote by η_p the cut-off at the level $p \in \mathbb{R}^+$

$$\begin{aligned} \eta_p(s) &= s \text{ if } |s| \leqslant p, \\ &= sign(s)\, p \text{ otherwise.} \end{aligned}$$

Definition 5.1. For any local functional $F(w)$, we define $\hat{F}(\rho) = E_{\mu_\rho}[F]$ where μ_ρ is the unique Gibbs measure with density $\rho \in W$.

By the virial theorem (see the appendix of Varadhan (1991)):

$$\hat{G}_k^{\alpha\sigma} = 2\delta_{\alpha\sigma}(P(\rho,\beta) - \beta^{-1}\rho).$$

We consider a measurable function $m : \mathbb{T}^d \to \mathbb{R}^+$ and we define the following functional:

$$\Omega(s,z,m) = \left[\dot{\lambda}(s,z) - \frac{2}{\gamma}\,|\nabla\lambda(s,z)|^2\right] m(z) - \frac{2}{\gamma}\,\Delta\lambda(s,z)P(m(z)). \tag{5.18}$$

Observing that $\partial_\rho\lambda(\rho) = 1/\rho\partial_\rho P(\rho)$, we get by integration by parts:

$$\int_{\mathbb{T}^d} \mathrm{d}z\, P(\rho(s,z))\Delta\lambda(s,z) = -\int_{\mathbb{T}^d} \mathrm{d}z\rho(s,z)|\nabla\lambda(s,z)|^2.$$

Therefore, we can replace $\dot{\lambda}(s,z)\rho(s,z)$ by $\Omega(s,z,\rho(s,\cdot))$ in (5.17)

$$\limsup_{k\to+\infty}\limsup_{\varepsilon\to 0} \int_0^t \left\{\varepsilon^d\,\frac{\mathrm{d}H_\varepsilon(s)}{\mathrm{d}s} + E_{f_\varepsilon^s}\left[\int_{\mathbb{T}^d} \mathrm{d}z\beta\,(J_k(s,z) - \Omega(s,z,\rho))\right]\right\}\mathrm{d}s \leqslant 0 \tag{5.19}$$

Now, we decompose $J_k(s,z) - \Omega(s,z,\rho)$ as follows:

$$J_k(s,z) - \Omega(s,z,\rho) = \sum_{l=1}^{4} \Omega_k^l(s,z)$$

with

$$\Omega_k^1(s,z) = [\Omega(s,z,\Theta_k(\omega_\varepsilon^z)) - \Omega(s,z,\rho(s,z))]\sigma_k(\omega_\varepsilon^z),$$

$$\Omega_k^2(s,z) = [J_k^{(p)}(s,z) - \Omega(s,z,\Theta_k(\omega_\varepsilon^z))]\sigma_k(\omega_\varepsilon^z),$$

$$\Omega_k^3(s,z) = [J_k^{(p)}(s,z) - \Omega(s,z,\rho(s,z))](1 - \sigma_k(\omega_\varepsilon^z)),$$

$$\Omega_k^4(s,z) = [J_k(s,z) - J_k^{(p)}(s,z)],$$

where $J_k^{(p)}(s,z)$ is $J_k(s,z)$ with $(\eta_p o G^{\alpha\sigma})_k$ instead of $G_k^{\alpha\sigma}$.

The next estimates state that the error we made from introducing the cut-off is negligible. From theorem indexed (3.21) in Butta and Lebowitz (1999), we have:

Proposition 5.2. *There exists a constant* κ_0 *such that, for* $l = 3$ *and* 4:

$$\limsup_{p\to\infty} \limsup_{k\to\infty} \limsup_{\varepsilon\to 0} \sup_{s\geqslant 0} \left\{ E_{f_\varepsilon^s}\left[\int_{\mathbb{T}^d} \mathrm{d}z \beta\Omega_k^l(s,z)\right] - \kappa_0^{-1} H_\varepsilon(s)\right\} \leqslant 0.$$

Moreover, using the local ergodic theorem, see the next section, for $\Omega_k^2(s,z)$, we get

$$\limsup_{p\to\infty} \limsup_{k\to\infty} \limsup_{\varepsilon\to 0} \int_0^t \mathrm{d}s E_{f_\varepsilon^s}\left[\int_{\mathbb{T}^d} \mathrm{d}z |\beta\Omega_k^2(s,z)|\right] = 0. \tag{5.20}$$

Conclusion of the step 3: Finally, from (5.19), we get that for all $t \in \mathbb{R}^+$

$$\varepsilon^d H_\varepsilon(t) + \int_0^t \mathrm{d}s \left\{ E_{f_\varepsilon^s}\left[\int_{\mathbb{T}^d} \mathrm{d}z \beta\Omega_k^1(s,z)\right] - 2\kappa_0^{-1} H_\varepsilon(s)\right\} \leqslant \mathrm{o}(\varepsilon,k), \tag{5.21}$$

where

$$\limsup_{k\to+\infty} \limsup_{\varepsilon\to 0} \mathrm{o}(\varepsilon,k) = 0.$$

Step 4: It consists on using the Gronwald's lemma and the Large Deviation Principle theorem (see inequality (3.25) of Butta and Lebowitz (1999)).

By (3.1), we have for every $\kappa > 0$ and for $s \in [0,t]$:

$$E_{f_\varepsilon^s}\left[\int_{\mathbb{T}^d} \mathrm{d}z \beta\Omega_k^1(s,z)\right] \geqslant -\kappa^{-1} H_\varepsilon(s) - \kappa^{-1}\varepsilon^d \log E_{\psi_\varepsilon^s}\left[\exp\left(-\kappa\varepsilon^{-d} \int_{\mathbb{T}^d} \mathrm{d}z \beta\Omega_k^1(s,z)\right)\right]$$

We integrate and then we apply the Gronwall lemma and we get from (5.21) and from the above inequality, that for any $t \in \mathbb{R}^+$

$$\varepsilon^d H_\varepsilon(t) \leqslant \mathrm{e}^{(\kappa^{-1}+2\kappa_0^{-1})t} \left(o(\varepsilon,k) + \kappa^{-1}\varepsilon^d \int_0^t \mathrm{d}s \log E_{\psi_\varepsilon^s}\left[\exp\left(-\kappa\varepsilon^{-d} \int_{\mathbb{T}^d} \mathrm{d}z \beta\Omega_k^1(s,z)\right)\right]\right). \tag{5.22}$$

Now, we use the following Large Deviation Principle theorem.

Theorem 5.3.

$$\limsup_{k\to\infty}\limsup_{\varepsilon\to 0}\kappa^{-1}\varepsilon^d \log E_{\psi^s_\varepsilon}\left[\exp\left(-\kappa\varepsilon^{-d}\int_{\mathbb{T}^d} \mathrm{d}z\beta\Omega^1_k(s,z)\right)\right] \leqslant \kappa^{-1}\mathscr{L}_\kappa(s,\lambda)$$

with

$$\mathscr{L}_\kappa(s,\lambda) := \sup_m \left[\int_{\mathbb{T}^d} \mathrm{d}z(\kappa\beta[\Omega(s,z,m)-\Omega(s,z,\rho(s,\cdot))]1_{\mathscr{C}}(m(z)) \right.$$
$$\left. - I_\beta(\lambda(s,z),m(z)))\right], \tag{5.23}$$

where $m:,\mathbb{T}^d\to\mathbb{R}^+$ *integrable and*

$$I_\beta(\lambda,m) = \tilde{P}(\lambda,\beta)+a(m,\beta)-\lambda m$$

(see Section 5 of (Butta and Lebowitz (1999) for a proof).

Thus, from (5.22), applying the previous theorem,

$$\limsup_{\varepsilon\to 0}\varepsilon^d H_\varepsilon(t) \leqslant \mathrm{e}^{(\kappa^{-1}+2\gamma^{-1})t}\int_0^t \mathrm{d}s\,\frac{1}{\kappa}\,\mathscr{L}_\kappa(s,\lambda).$$

Up to this point, we only have to prove that for any κ sufficiently small $\mathscr{L}_\kappa(s,\lambda)=0$ for $s\in[0,t]$.

To see this, we firstly note that

$$-\int_{\mathbb{T}^d} \mathrm{d}z I_\beta(\lambda(s,z),m(z)) \leqslant 0$$
$$= 0 \text{ iff } m(z)=\rho(s,z)$$

because for $\lambda\in U$ fixed, $m\to I_\beta(\lambda,m)$ is a strictly convex function on $\mathscr{C}$, non-negative and equal to 0 iff $m=\nabla_\lambda P$,

Moreover, the functional $f(m) = \int_{\mathbb{T}^d}\mathrm{d}z\beta[\Omega(s,z,m)-\Omega(s,z,\rho(z))]1_{\mathscr{C}}(m(z))$ is bounded on the class of functions considered in (5.23) and equal to 0 if *for all* $z\in\mathbb{T}^d$, $m(z)=\rho(s,z)$.

Finally, for κ small enough, $\mathscr{L}_\kappa(s,\lambda)=0$ if $\partial f/\partial m(\rho(s,\cdot))=0$.

(Observe that $\rho(s,z)$ is away from $\mathbb{R}^+\setminus\mathscr{C}$ because $\mathrm{d}(K,\mathbb{R}^+\setminus\mathscr{C})>\eta/2$).

But, for any $(s,z)\in\mathbb{R}^+\times\mathbb{T}^d$, we have that

$$\beta^{-1}\frac{\partial f}{\partial m(z)}(\rho(s,\cdot)) = \dot{\lambda}(s,z)-\frac{2}{\gamma}[|\nabla\lambda(s,z)|^2+\Delta\lambda(s,z)P_\rho(\rho(s,z))].$$

Since $\partial_\rho\lambda(\rho)=1/\rho\partial_\rho P(\rho)$, we have that

$$|\nabla\lambda(s,z)|^2+\Delta\lambda(s,z)P_\rho(\rho(s,z))$$

$$= \frac{\partial \lambda(\rho(s,z))}{\partial \rho} \left[\frac{\partial \rho(s,z)}{\partial z} \nabla \lambda(s,z) + \rho(s,z) \Delta \lambda(s,z) \right]$$

$$= \frac{\partial \lambda(\rho(s,z))}{\partial \rho} \left[\frac{\partial}{\partial z} (\rho(s,z) \nabla \lambda(s,z)) \right].$$

Therefore

$$\beta^{-1} \frac{\partial f}{\partial m(z)} (\rho(s,\cdot)) = \frac{\partial \lambda(\rho(s,z))}{\partial \rho} \left[\frac{\partial \rho(s,z)}{\partial t} - \frac{2}{\gamma} \frac{\partial^2 P(\rho(s,z))}{\partial z^2} \right].$$

Because ρ satisfies Eq. (2.3), we finally get that:

$$\frac{\partial f}{\partial m(z)} (\rho(s,\cdot)) = 0.$$

Conclusion: If ρ satisfies Eq. (2.3), $\lim_{\varepsilon \to 0} \varepsilon^d H_\varepsilon(t) = 0$ and the proof of the Theorem 3.1 is finished. □

Appendix A. The local ergodic theorem

In this section, we sketch the proof of (5.20). Using the fact that $\partial_{\alpha\sigma} \lambda$ is a bounded function on $\mathbb{T}^d$, the proof of (5.20) can be reduced to prove

$$\limsup_{p \to +\infty} \limsup_{k \to +\infty} \limsup_{\varepsilon \to 0}$$

$$\int_0^t \mathrm{d}t E_{f_\varepsilon^t} \left[\int_{\mathbb{T}^d} \mathrm{d}z |(\eta_p o G^{\alpha\sigma})_k(\omega_\varepsilon^z) - \hat{G}^{\alpha\sigma}(\Theta_k(\omega_\varepsilon^z))| \sigma_k(\omega_\varepsilon^z) \right] = 0. \tag{A.1}$$

To show (A.1), it is enough to prove the following local ergodic theorem.

Theorem A.1. *Let F a local function, continuous and bounded on $\mathbb{T}^d$. Then,*

$$\lim_{k \to +\infty} \limsup_{\varepsilon \to 0} \int_0^t \mathrm{d}t E_{f_\varepsilon^t} \left[\int_{\mathbb{T}^d} |F(\omega_\varepsilon^z) - \hat{F}(\Theta_k(\omega_\varepsilon^z))| \sigma_k(\omega_\varepsilon^z) \, \mathrm{d}z \right] = 0.$$

Proof. The proof of this theorem is similar to the proof of Theorem 4.1 of Butta and Lebowitz (1999). So, we only give the outline.

Let us introduce

$$\bar{f}_\varepsilon^t(\underline{x}, \underline{v}) = \frac{1}{t} \int_0^t f_\varepsilon^s(\underline{x}, \underline{v}) \, \mathrm{d}s$$

and

$$\hat{f}_\varepsilon^t(\underline{x}, \underline{v}) = \int_{\mathbb{T}^d} \bar{f}_\varepsilon^t(\underline{x} + a, \underline{v}) \, \mathrm{d}a.$$

The theorem can be reduced to the proof of

$$\limsup_{k \to +\infty} \limsup_{\varepsilon \to 0} E_{\hat{f}_\varepsilon} [|F_k(\omega_\varepsilon^0) - \hat{F}(\Theta_k(\omega_\varepsilon^0))| \sigma_k(\omega_\varepsilon^0)] = 0. \tag{A.2}$$

Let $k_F \in \mathbb{N}$ be such that $supp\, F \subset D_{k_F}$ (we defined D_k in the step 2 of the last section). Let $\bar{k} = k + k_F$.

Consider the map $\mathbb{T}^{dN} \rightarrow \Omega$, $\underline{x} \mapsto \omega_\varepsilon^0$. For ε small enough, we can define the projection

$$\Pi_{\bar{k}} : \mathbb{T}^{dN} \rightarrow \Omega|_{D_{\bar{k}}}$$
$$\underline{x} \mapsto \Pi_{\bar{k}}(\underline{x}) = \omega_\varepsilon^0|_{D_{\bar{k}}}.$$

Call $\mathscr{B}_{\bar{k}}$ the family of limit points (for the weak topology) of $\{v_\varepsilon = \Pi_{\bar{k}}(\hat{f}_\varepsilon d\mu_\varepsilon); \varepsilon \in (0,1]\}$.

- We prove that the family $\{v_\varepsilon = \Pi_{\bar{k}}(\hat{f}_\varepsilon \mathrm{d}\mu_\varepsilon); \varepsilon \in (0,1]\}$ is tight.
  Let us then introduce $\mu_{m,\bar{k}}^{\bar{\omega}}$ the canonical Gibbs measure on the cube $D_{\bar{k}}$ with boundary conditions $\bar{\omega} \in \Omega$ and number of particles m.
- We prove after that for any $v \in \mathscr{B}_{\bar{k}}$, v can be written as
  $$v(\mathrm{d}\omega) = \int \hat{v}(\mathrm{d}\hat{\omega}, \mathrm{d}m)\mu_{m,\bar{k}}^{\bar{\omega}}(\omega), \tag{A.3}$$
  where $\hat{v}(\mathrm{d}\hat{\omega}, \mathrm{d}m)$ is a measure supported on $\{m \leqslant D_{\bar{k}}\}$.
  The conclusion of the theorem is then exactly the same as in Butta and Lebowitz (1999). The only difference comes from the proof of (A.3).
- *Proof of* (A.3): It is, by a straightforward extension to higher dimensions of Lemma 4.5 of Olla and Varadhan (1991). The proof use the following remark:

Remark A.2. Let Λ be some finite subset on $\mathbb{R}^{Nd}$. Define

$$\tilde{I}_\varepsilon(\bar{f}_\varepsilon) := \int \sum_{i=1, \varepsilon^{-1} x_i \in \Lambda}^{N} \frac{1}{\bar{f}_\varepsilon(w)} |\nabla_{v_i} \bar{f}_\varepsilon(w)|^2 \, \mathrm{d}\mu_\varepsilon$$

then, there exists a constant C' such that

$$\tilde{I}_\varepsilon(\bar{f}_\varepsilon) \leqslant C'\varepsilon^2.$$

Proof. We use estimation (ii) of Lemma 4.1 and the convexity of the functional I_ε:

$$\varepsilon^d \int \sum_{i=1}^{N} \frac{1}{\bar{f}_\varepsilon(w)} |\nabla_{v_i} \bar{f}_\varepsilon(w)|^2 \, \mathrm{d}\mu_\varepsilon \leqslant C''\varepsilon^2.$$

Let us introduce a continuous non-positive function h with compact support denoted Λ such that $\int_{\mathbb{R}^d} h(z)\,\mathrm{d}z = 1$. From the above inequality, we get

$$\varepsilon^d \int \sum_{i=1}^{N} \int_{\mathbb{R}^d} \mathrm{d}z \; h(\varepsilon^{-1}x_i - z) \frac{1}{\bar{f}_\varepsilon(w)} |\nabla_{v_i} \bar{f}_\varepsilon(w)|^2 \mathrm{d}\mu_\varepsilon \leqslant C''\varepsilon^2.$$

Using translation invariance, the above expression is smaller than

$$\varepsilon^d \int_{(\varepsilon^{-1}\Lambda)^d} \mathrm{d}z \int \sum_{i=1}^{N} h(\varepsilon^{-1}x_i) \frac{1}{\bar{f}_\varepsilon(w)} |\nabla_{v_i} \bar{f}_\varepsilon(w)|^2 \mathrm{d}\mu_\varepsilon \leqslant C''\varepsilon^2.$$

Finally,

$$\int \sum_{i=1}^{N} h(\varepsilon^{-1}x_i)\frac{1}{\bar{f}_{\varepsilon}(w)}\,|\nabla_{v_i}\bar{f}_{\varepsilon}(w)|^2\,\mathrm{d}\mu_{\varepsilon} \leqslant C''\varepsilon^2. \qquad \square$$

References

Butta, P., Lebowitz, J.L., 1999. Hydrodynamic limit of Brownian particles interacting with short and long range forces. J. Statist. Phys. 94 (3–4), 653–694.

Kipnis, C., Landim, C., 1999. Scaling Limit of Interacting Particles Systems. Springer-Verlag, Berlin.

Olla, S., Varadhan, S.R.S., 1991. Scaling limits for interacting Ornstein–Uhlenbeck diffusions. Comm. Math. Phys. 135, 355–378.

Olla, S., Varadhan, S.R.S., Yau, H.T., 1993. Hydrodynamical limit for an Hamiltonian system with weak noise. Comm. Math. Phys. 135, 523–560.

Ruelle, D., 1969. Statistical Mechanics: Rigorous Results. W.A. Benjamin, Inc, New York, Amsterdam.

Ruelle, D., 1970. Superstable interactions in classical statistical mechanics. Comm. Math. Phys. 18, 127–159.

Spohn, H., 1991. Large Scale Dynamics of Interacting Particles, Text and Monographs in Physics. Springer-Verlag, Berlin.

Varadhan, S.R.S., 1991. Scaling limit for interacting diffusions. Comm. Math. Phys. 135, 313–353.

Yau, H.T., 1991. Relative entropy and hydrodynamics of Ginzburg–Landau models. Lett. Math. Phys. 22, 63–80.